Efficient Computation Of Option Greeks Using Malliavin Calculus

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Abstract

In this paper, we review an effective approach for computing option Greeks through the lens of Malliavin Calculus, a branch of Stochastic Calculus that enables the differentiation of random variables. This approach provides more stable and efficient results compared to the finite difference methods, especially for non-smooth payoff functions.

From Weierstrass to Malliavin

In the 19th century, mathematicians such as Augustin-Louis Cauchy and Karl Weierstrass pushed for a transformation in mathematical analysis that gave rigor and clarity to the previously vague concept of differentiability. While Cauchy defined the limit and formalized the notion of derivative, Weierstrass gave a full, rigorous definition and drew a clear distinction between continuity and differentiability.

Back in the 18th century, it was assumed that any continuous function should be differentiable. There were only a few isolated examples where continuity did not imply differentiability. Take, for example, the absolute value function $f(x) = |x|$, which is differentiable everywhere except for $x = 0$. However, in 1872, Karl Weierstrass presented an example of a continuous function that was not differentiable at any point:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$$

for some carefully chosen values of a and b. This pathological function was the first example of a continuous nowhere-differentiable function. It was at this moment that it became clear that continuity does not imply differentiability. In fact, it was discovered that most continuous functions are nowhere differentiable. It is really the differentiable functions that are the exceptions rather than the rule.

Surprisingly enough, around 50 years earlier, a Scottish botanist named Robert Brown had discovered another example of a continuous nowhere-differentiable function while studying pollen grains in water under a microscope. Brown noticed that tiny particles inside the pollen grains exhibited a continuous, erratic movement. This phenomenon

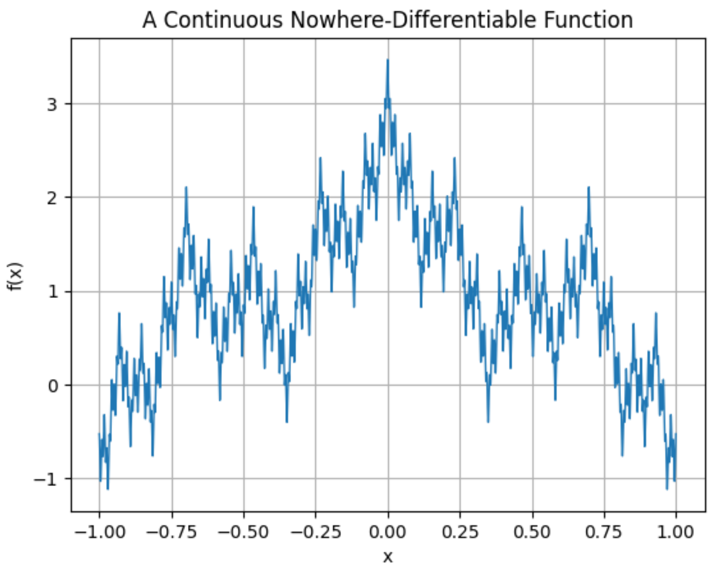

Figure 1: A sketch of Weierstrass function for $a = 3/4$ and $b = 3$.

would receive the name of Brownian Motion. Almost two centuries later, in 1905, Albert Einstein provided a theoretical explanation for Brownian motion in his paper on molecular kinetics. Einstein's mathematical model connected Brownian motion with diffusion equations.

Later, in the 1920s, an American mathematician called Norbert Wiener gave a mathematically rigorous definition of Brownian Motion. Wiener modeled this phenomenon as a stochastic process and defined Brownian Motion as a continuous-time random walk with the following properties:

- Independent, stationary, and normally distributed increments.
- Continuous (but nowhere differentiable) paths.
- Starts at zero.

The non-differentiability nature of the process is due to its fractal behavior. The erratic oscillations are present at every scale, no matter how much we "zoom in" on a Brownian Motion path. An informal proof of this non-differentiability property is the following:

Consider a Brownian motion W_t. The increments $W_{t+h} - W_t$ are distributed normally with zero mean and standard deviation $\sqrt{h}$. If we consider the following difference quotient:

$$\frac{W_{t+h} - W_t}{h} \sim \frac{\sqrt{h}}{h}\mathcal{N}(0,1) = \frac{\mathcal{N}(0,1)}{\sqrt{h}} \to \infty \quad \text{as } h \to 0$$

In other words, if we try to compute the derivative of W_t, we get that it fluctuates between $-\infty$ and ∞. In order to illustrate how "wild" these oscillations are, we provide some non-differentiable facts about Brownian Motion:

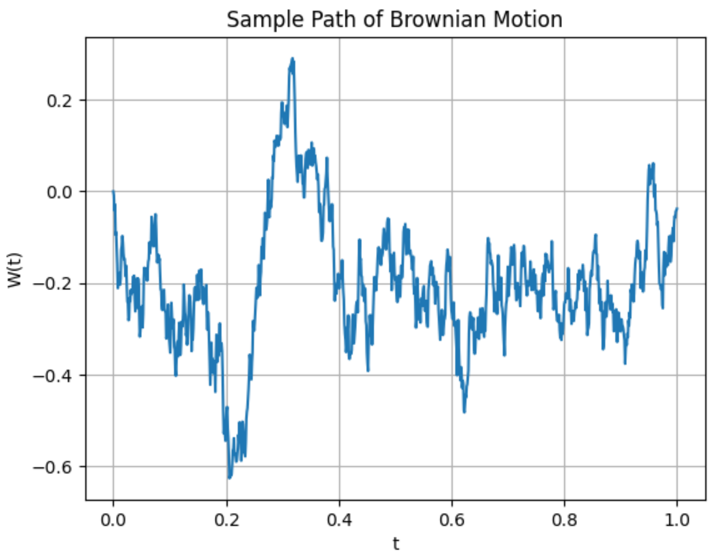

Figure 2: Sample path of a Brownian Motion

- For a Brownian Motion starting at 0, the probability that it crosses zero infinitely many times in any interval $(0, t)$ is 1.
- For a Brownian Motion starting at 0, the probability that it crosses zero in the interval (t_1, t_2), where $0 < t_1 < t_2$, is strictly greater than zero.

After defining Brownian Motion rigorously, another challenge arose in the world of Stochastic Calculus. The question was whether it was possible to make sense of the following integral:

$$\int f\, dW$$

Intuitively, a conventional Riemann integral $\int f\, dx$ can be defined as the sum of infinitely many rectangles of width dx. However, how can we define an integral where the width of the rectangles, dW, is random? Moreover, it is impossible to construct a Lebesgue-Stieltjes integral of the form:

$$\int f(x)\, dg = \int f\, g'(x) dx$$

if g is nowhere differentiable. The problem remained unsolved for 30 years. However, in the late 1940s, Japanese mathematician Kiyoshi Itô gave a formal definition to this integral in a probabilistic manner. This integral would receive the name of Itô Integral and is defined as:

$$\int_0^T f_t\, dW_t = \lim_{N\to\infty} \sum_{j=1}^{N} f_{t_{j-1}}(W_{t_j} - W_{t_{j-1}})$$

where $t_j = jT/N$ and f_t is an adapted process which is square integrable. Additionally, Itô proved an equivalent version of the chain rule for Stochastic Calculus, named Itô's Lemma:

$$df = \frac{\partial f}{\partial X_t} dX_t + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (dX_t)^2$$

Here, f is a twice differentiable function and X_t is an Itô Process, a special type of stochastic process that is a function of both time and Brownian Motion, and is composed of a drift term and a diffusion term.

We now return to the Itô Integral. After defining this integral, a natural question arises: is there an explicit expression for the integrand? In other words, given an Itô integral of the form:

$$M_t = \int_0^t f_s dW_s$$

can we define f_s as a kind of differentiation operator within a probabilistic framework? This is, in fact, made possible by the Malliavin Derivative. This operator was introduced by the French mathematician Paul Malliavin in his 1976 paper "Stochastic Calculus of Variation and Hypoelliptic Operators", which laid the foundation for a new branch of Stochastic Analysis known as Malliavin Calculus.

The Malliavin Derivative allows us to compute derivatives of random variables and stochastic processes. It is, in fact, a way to differentiate continuous nowhere-differentiable functions, although it is not a conventional derivative (defined with a limit) but a stochastic derivative that mimics the properties of classical differentiation.

Finally, in 1999, the Malliavin Calculus found an interesting application in the world of mathematical finance, the computation of option Greeks, which are the derivatives of option prices with respect to one or more underlying parameters. As we will prove later, this new approach is more accurate than traditional finite difference methods, especially for exotic options with non-smooth payoffs.

For a more in-depth explanation of the history of Malliavin Calculus, we refer to [4].

A New Approach For Computing Greeks

In this section, we shift our focus from Stochastic Calculus to its applications in finance. Suppose we are a financial institution interested in selling option contracts. An option is a financial contract that gives the buyer the right, but not the obligation, to buy or sell an asset at a specific price (strike) on or before a specific date (maturity).

A financial institution that sells an option wants to protect itself from potential losses. For example, if it sells a call option (which gives the buyer the right to buy a stock), and the stock price rises above the strike price, the institution will have to sell the stock at a lower price than the market. To reduce this risk, the institution holds some shares of the stock. These shares help offset losses if the stock goes up. However, if the stock stays below the strike price, holding shares could lead to losses instead.

That is why the institution needs to actively manage its stock holdings, buying or selling shares as the stock price changes, to keep its risk under control. The value of an

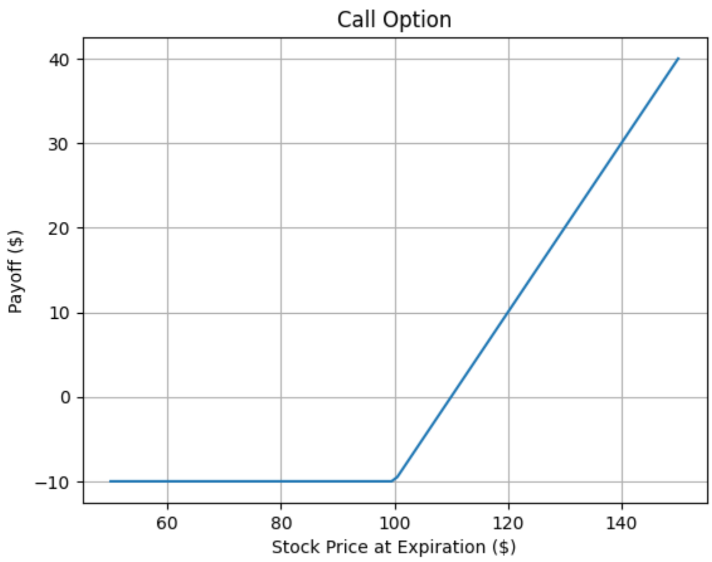

Figure 3: Call option payoff at maturity

option is influenced not only by the stock price but also by factors such as time to maturity, implied volatility, and interest rates. The sensitivities of the option's value to these factors are known as the Greeks, and they play a crucial role in hedging the associated risks. For our analysis, we will focus primarily on the risks stemming from stock price movements, as they are typically the most significant source of risk for an option's value.

If we denote the option value as V and the stock price as S, we can express the variations on the value of this contract with respect to the changes in the underlying stock using a Taylor expansion:

$$\Delta V \approx \frac{\partial V}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\Delta S)^2 + \cdots$$

Where:

$$\Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}$$

These sensitivities are called Delta and Gamma. Delta tells us how many shares the institution should hold to protect against small changes in the stock price. Gamma measures how quickly Delta changes when the stock price moves. It helps explain the error that happens when we hedge using only Delta, especially during larger price movements.

It is clear that computing Delta and Gamma is crucial for successfully hedging an options portfolio. If the option is vanilla (a standard, plain option with no special or complex features), these Greeks can be computed analytically in the Black-Scholes framework with the following formulas:

$$\Delta = \Phi(d_+), \quad \Gamma = \frac{\phi(d_+)}{S\sigma\sqrt{T}}$$

Where Φ is the standard cumulative normal distribution, ϕ is the standard normal distribution, σ is the implied volatility, T is the time to maturity, K is the strike price and d_+ is given by:

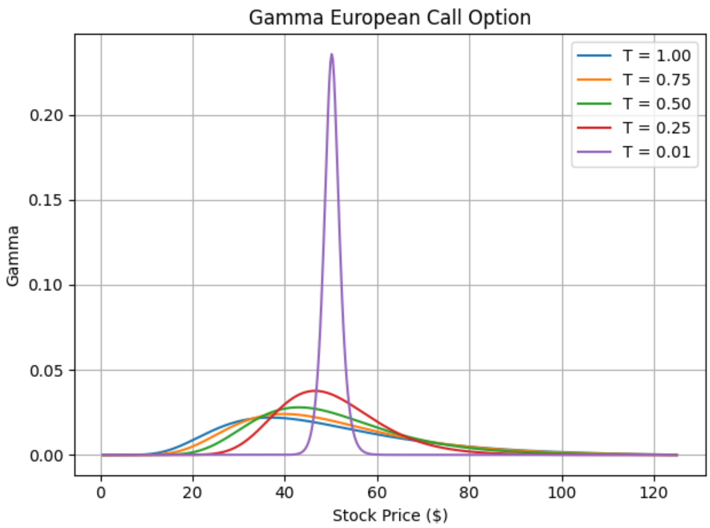

Figure 4: Gamma of a vanilla option for different time values

$$d_+ = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

However, what happens when we are dealing with an exotic option, one with more complex features than a vanilla option? The short answer is that there is no closed-form solution, so we will need to approximate the Greeks using a combination of a simulation of a random variable for computing the option price and a finite difference approach for calculating the derivatives.

For the random variable simulation, we need to assume a price dynamic for the underlying asset. In the simplest setup, it is often considered that the stock price follows a Geometric Brownian Motion, with the following expression:

$$S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right]$$

Here, we can express the Brownian Motion as:

$$W_T = \sqrt{T}Z, \quad Z \sim \mathcal{N}(0,1)$$

Once we have simulated the stock price, we can calculate the value of the option for an arbitrary payoff function f as follows:

$$V = e^{-rT}\,\mathbb{E}\left[f(S_T)\right]$$

Where $\mathbb{E}$ denotes the expectation under the risk-neutral probability measure. Finally, we compute the Greeks with a finite difference approach:

$$\Delta \approx \frac{V(S_T + \Delta S) - V(S_T)}{\Delta S}, \quad \Gamma \approx \frac{V(S_T + \Delta S) - 2V(S_T) + V(S_T - \Delta S)}{(\Delta S)^2}$$

This approach introduces two sources of numerical error. The first is due to the simulation of S_T, where we need to draw samples from a standard normal distribution.

Since we cannot take an infinite number of samples, this introduces a slight error in the calculation of the option price.

The second, more significant source of error arises from the finite difference method, which depends on the step size ΔS and the smoothness of the payoff function f. Choosing an appropriate ΔS minimizes the error in approximating the derivatives. However, if the payoff function f has jumps or discontinuities, it becomes much harder to improve the accuracy of the computation, as the derivatives at those points tend to infinity. Take, for example, a digital call option, whose payoff is the following:

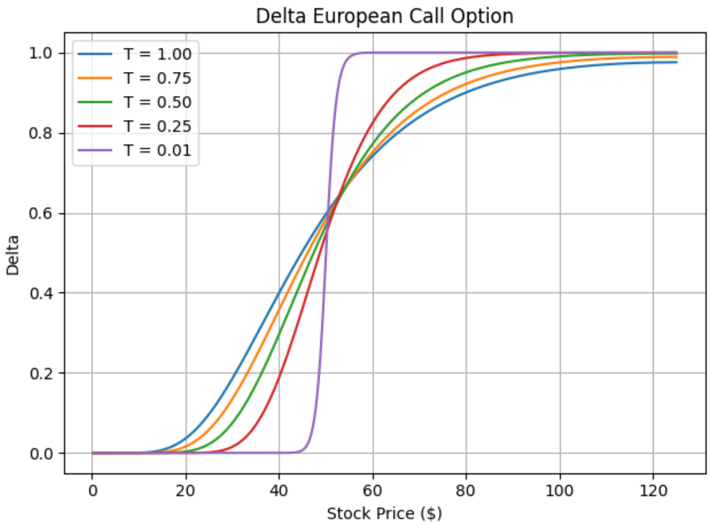

Figure 5: Digital call option for various time values

If we try to compute the derivatives numerically at the strike (where the option jumps), we will have large precision issues. This is where Malliavin Calculus can significantly outperform the finite difference approach. As we will explore in the next section, we can compute any derivative of the option value V as:

$$\frac{\partial V}{\partial x} = e^{-rT}\,\mathbb{E}\left[f(S_T)\cdot weight\right]$$

where x is any underlying parameter of the option (stock price, volatility, etc.). In other words, we have to multiply the payoff by a certain weight function that makes the option price formula match its derivatives. With this approach, we completely eliminate the numerical error that stems from the finite difference calculation.

The concepts from Malliavin Calculus allow us to find this weight function for any payoff function, no matter how vanilla or exotic the option is. For a deeper look at option hedging strategies, the reader can check [3].

Tools From Malliavin Calculus

We now move to a more formal section where we will define the mathematical tools required for deducing the weight function shown previously. We begin by defining the

Malliavin Derivative, which is an operator that enables us to differentiate random variables.

Let us first recall the definition of the chain rule for a deterministic function. For a smooth function $f(x_1, \ldots, x_n)$ and deterministic variables x_i, the derivative of f with respect to a parameter t (if each x_i depends on t) is:

$$\frac{d}{dt} f(x_1(t), \ldots, x_n(t)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{dx_i}{dt}$$

The Malliavin Derivative can be viewed as the stochastic case of this formula.

Definition 1 (Malliavin Derivative) *Let F be a square-integrable random variable of the form:*

$$F = f(W_{t_1}, W_{t_2}, \ldots, W_{t_n}),$$

Where f is a smooth, deterministic function. The Malliavin Derivative of F, denoted as DF is defined as follows:

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, \ldots, W_{t_n}) \cdot \mathbf{1}_{[0,t_i]}(t) \tag{1}$$

Throughout the remainder of the text, we will use D and D_t as equivalent notations. We now give some examples:

Example 1 (Malliavin Derivative of Brownian Motion) *Consider the random variable $F = W_t$ and the function $f(x) = x$. Then, using (1) we have that:*

$$DW_t = f'(W_t) \cdot \mathbf{1}_{[0,t_i]}(t) = \mathbf{1}_{[0,t_i]}(t)$$

This, intuitively, makes sense. In this example, we are performing a differentiation-type operation where we can understand the Malliavin Derivative as something like $D = d/dW_t$. Thus, it makes sense that the derivative of W_t with respect to itself is one.

Example 2 (Malliavin Derivative of Geometric Brownian Motion) *Consider the random variable $F = S_t$ and the function:*

$$f(x) = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right) T + \sigma x\right]$$

So that we have:

$$S_t = f(W_t) = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right) T + \sigma W_t\right]$$

Then, using (1) we get that:

$$D_s S_t = \sigma\, S_t\, \mathbf{1}_{[0,t]}(s)$$

Again, this makes sense. If we differentiate S_t with respect to W_t we should get the σ value down just as when we differentiate an exponential function. Note that we are not differentiating in a conventional way with a limit, but in a stochastic way that mimics the deterministic differentiation.

The next concept we will be defining is the Skorokhod Integral, which is a generalization of the Itô Integral that allows integration of non-adapted processes. It is also the adjoint operator of the Malliavin Derivative, linking differentiation and integration in the Malliavin Calculus framework in a way that preserves inner products.

Definition 2 (Skorokhod Integral) *For a process $u \in Dom(\delta)$, the Skorokhod Integral is denoted by:*

$$\delta(u) = \int_0^T u_t \, \delta W_t \tag{2}$$

where $u \in Dom(\delta)$ is the subset of square-integrable processes for which $\delta(u)$ exists and is also square-integrable.

If the process u_t is adapted, then the Skorokhod Integral is a conventional Itô Integral:

$$\delta(u) = \int_0^T u_t \, dW_t$$

Equipped with integral and differentiation operations, we are now ready to define the stochastic version of the integration by parts (IBP) formula. From ordinary calculus, we know that:

$$\int f \, dg = fg - \int g \, df$$

The following definition gives a stochastic equivalent of this rule:

Definition 3 (Malliavin IBP Formula) *Consider a random variable F and a square integrable process u, such that u and $Fu \in Dom(\delta)$. Then:*

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle \tag{3}$$

where $\langle DA, u \rangle$ denotes the inner product in the space of square-integrable functions, so:

$$\langle DF, u \rangle = \int_0^T D_t F \cdot u_t \, dt$$

A useful trick that we will use later on is setting $u = 1$ so we get:

$$\delta(F) = FW_T - \int_0^T D_t F \, dt \tag{4}$$

Note that $\delta(1)$ is an Itô Integral so we have $\delta(1) = \int_0^T dW_t = W_T$. With this, we can easily compute the Skorokhod Integral of random variables if we know their Malliavin Derivative. The following examples illustrate this:

Example 3 (Skorokhod Integral of Brownian Motion) *Consider the random variable $F = W_T$. Using (4) we have that:*

$$\delta(W_T) = W_T^2 - \int_0^T D_t W_T \, dt = W_T^2 - T$$

Let us now try with Geometric Brownian Motion:

Example 4 (Skorokhod Integral of Geometric Brownian Motion) *For $F = S_T$, we get:*

$$\delta(S_T) = S_T \, W_T - \int_0^T D_t S_T \, dt = S_T W_T - \sigma S_T T$$

We are now ready to define the formula for computing the weight function mentioned in the previous section. For this purpose, we will use another integration by parts formula that involves expectations of random variables. As we will see later, the Skorokhod Integral present in this formula is the weight function we have been looking for.

Definition 4 (Malliavin IBP Formula for Expectations) *Consider two random variables F and G and a random process u, where F is Malliavin Differentiable and $u \in Dom(\delta)$, then:*

$$\mathbb{E}\big[f'(F)G\big] = \mathbb{E}\left[f(F) \, \delta\left(\frac{Gu}{\langle DF, u\rangle}\right)\right] \tag{5}$$

We conclude this chapter with the following example:

Example 5 (Expectation of the Brownian Motion Malliavin Derivative) *Consider $F = W_T$, $G = 1$ and $u = \mathbf{1}_{[0,T]}$. We can use (5) to derive the following:*

$$\mathbb{E}\big[f'(W_T)\big] = \mathbb{E}\left[f(W_T) \, \delta\left(\frac{1}{T}\right)\right] = \frac{1}{T}\mathbb{E}\big[f(W_T)W_T\big]$$

Where $\langle DW_T, u\rangle = T$ and $\delta(1/T) = W_T/T$.

Note that we have computed the derivative of a function of Brownian Motion by multiplying the original function $f(W_T)$ by a certain weight given by $\delta(1/T)$. This is almost identical to what we have discussed in the previous section. The next step will be to use this formula to compute Delta and Gamma.

This section follows the exposition in [2]. For a more rigorous and extensive treatment of Malliavin calculus, we refer the reader to that reference.

Greeks Computation

In this section, we will compute Delta and Gamma using the Malliavin Calculus techniques, following [2] closely. After that, we will perform some numerical experiments for vanilla and exotic options. Recall that we are looking for a weight function that satisfies:

$$\frac{\partial V}{\partial x} = e^{-rT} \, \mathbb{E}\left[f(S_T) \cdot weight\right]$$

In our case, we are interested in computing Delta as:

$$\Delta = \frac{\partial V}{\partial S} = e^{-rT}\,\mathbb{E}\left[f(S_T)\cdot weight_\Delta\right]$$

and Gamma:

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = e^{-rT}\,\mathbb{E}\left[f(S_T)\cdot weight_\Gamma\right]$$

From our previous discussion, we know that this weight function is given by a Skorokhod Integral, thus:

$$\Delta = e^{-rT}\,\mathbb{E}\left[f(S_T)\cdot\delta(w_\Delta)\right], \quad \Gamma = e^{-rT}\,\mathbb{E}\left[f(S_T)\cdot\delta(w_\Gamma)\right]$$

We can derive the expressions for $\delta(w_\Delta)$ and $\delta(w_\Gamma)$ using the tools defined in the previous section. From now on, we will use the notation S_0 instead of S to denote the stock price. Let us start by computing Delta:

$$\Delta = \frac{\partial}{\partial S_0}e^{-rT}\,\mathbb{E}\left[f(S_T)\right] = e^{-rT}\,\mathbb{E}\left[\frac{\partial}{\partial S_0}f(S_T)\frac{\partial S_T}{\partial S_0}\right] = e^{-rT}\,\mathbb{E}\left[f'(S_T)\frac{S_T}{S_0}\right]$$

Note that $S_T = f(S_0)$ so we have to apply the chain rule and that $\partial S_T/\partial S_0 = S_T/S_0$. If we take out S_0 from the expectation, we get:

$$\Delta = \frac{e^{-rT}}{S_0}\,\mathbb{E}\left[f'(S_T)S_T\right]$$

where we can apply (5) assuming $G = S_T$, $F = S_T$ and $u = \mathbf{1}_{[0,T]}$. Therefore:

$$\Delta = \frac{e^{-rT}}{S_0}\,\mathbb{E}\left[f(S_T)\delta\left(\frac{S_T}{\sigma S_T T}\right)\right] = \frac{e^{-rT}}{S_0 T\sigma}\,\mathbb{E}\left[f(S_T)W_T\right]$$

Here we have computed $\langle DS_T, u\rangle = \sigma T S_T$ (see example 2) and the Skorokhod Integral is an Itô Integral as there are no random variables involved. Thus, the expression for the Delta for any option with payoff f is the following:

$$\boxed{\Delta = \frac{e^{-rT}}{S_0 T\sigma}\,\mathbb{E}\left[f(S_T)W_T\right]} \tag{6}$$

We now proceed to the calculation of Gamma:

$$\Gamma = \frac{\partial^2}{\partial S_0^2}e^{-rT}\,\mathbb{E}\left[f(S_T)\right] = e^{-rT}\frac{\partial}{\partial S_0}\mathbb{E}\left[f'(S_T)\frac{S_T}{S_0}\right] = e^{-rT}\mathbb{E}\left[f''(S_T)\left(\frac{S_T}{S_0}\right)^2\right]$$

Note that S_T/S_0 does not depend on S_0 here. Next we have that:

$$\Gamma = \frac{e^{-rT}}{S_0^2}\mathbb{E}\left[f''(S_T)S_T^2\right]$$

and using (5) where $G = S_T^2$, $F = S_T$ and $u = \mathbf{1}_{[0,T]}$ we get the following:

$$\Gamma = \frac{e^{-rT}}{S_0^2}\mathbb{E}\left[f'(S_T)\delta\left(\frac{S_T^2}{\sigma T S_T}\right)\right] = \frac{e^{-rT}}{S_0^2 T\sigma}\mathbb{E}\left[f'(S_T)\delta(S_T)\right]$$

From example 5 we now that $\delta(S_T) = S_T W_T - \sigma S_T T$, so:

$$\Gamma = \frac{e^{-rT}}{S_0^2 T\sigma}\mathbb{E}\left[f'(S_T)(S_T W_T - \sigma S_T T)\right]$$

and using (5) again but setting $G = S_T W_T - \sigma S_T T$ we reach the expression below:

$$\Gamma = \frac{e^{-rT}}{S_0^2 T\sigma}\mathbb{E}\left[f(S_T)\delta\left(\frac{S_T W_T - \sigma S_T T}{\sigma T S_T}\right)\right] = \frac{e^{-rT}}{(S_0 T\sigma)^2}\mathbb{E}\left[f(S_T)\delta(W_T - \sigma T)\right]$$

We can split this last integral into a Skorokhod Integral $\delta(W_T)$ and an Itô Integral $\delta(T\sigma)$, and using the previous examples, we can reach the final expression of Gamma:

$$\boxed{\Gamma = \frac{e^{-rT}}{(S_0 T\sigma)^2}\mathbb{E}\left[f(S_T)(W_T^2 - T - W_T\sigma T)\right]} \tag{7}$$

We are now ready to perform some numerical experiments to test the accuracy of the Malliavin approach. Consider a European call option with the following parameters:

Volatility (σ)	Time to Maturity (T)	Strike Price (K)	Risk-Free Rate (r)
0.45	1 year	\$50	1%

Table 1: Parameters European call option

If we compute the Delta for several values of S, we get that the Malliavin Delta matches exactly the finite difference approximation and the analytical solution.

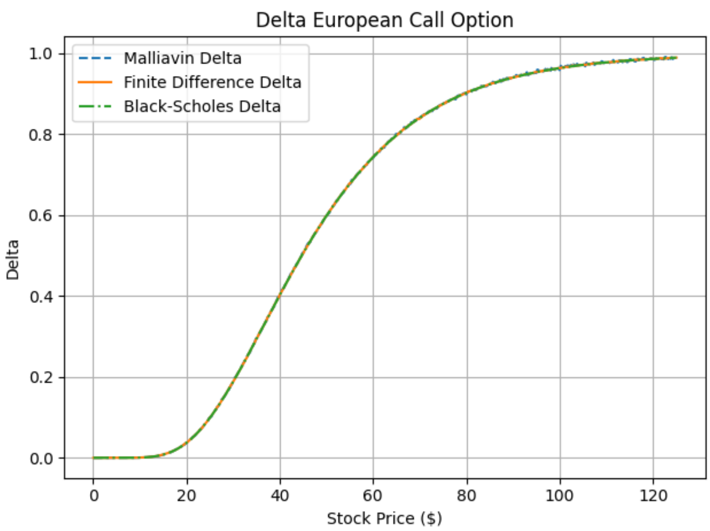

Figure 6: Delta of a European call option

It makes sense to have a closer look at $S = K$, where the numerical issues are more present, as this is the point where the Delta jumps, especially when the option is close to maturity. Performing various simulations at the strike price, we get:

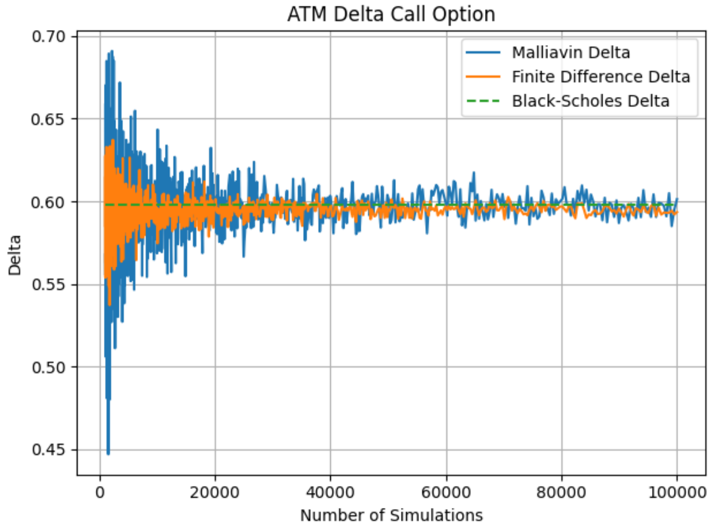

Figure 7: ATM Delta of a European call option

As we can see, both the Malliavin and the finite difference method match the theoretical value of the Delta. However, the Malliavin Delta is much noisier, so despite estimating correctly the value, it is not superior to the finite difference approach.

Now we will have a look at a more pathological case where the payoff function is non-smooth. Consider a digital call option whose payoff at maturity is given by:

$$f(S_T) = \begin{cases} 1 & \text{if } S_T \leq K \\ 0 & \text{if } S_T > K \end{cases}$$

If we compute the Delta for various values of S, for the parameters in Table 1, we get:

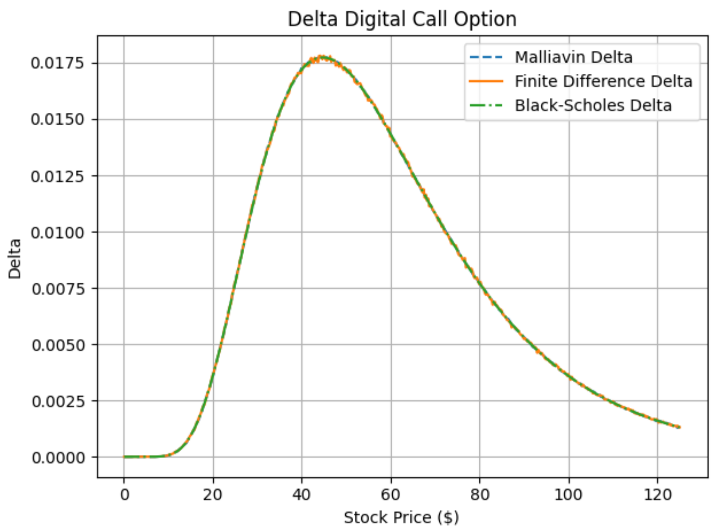

Figure 8: Delta of a digital call option

Here, it is clear that we will have some difficulties at $S = K$, and if we look closer at the graph, we can see that the finite difference method gets noisier around that point. If we zoom in on the strike and test the two methods for different numbers of simulations, we obtain the following:

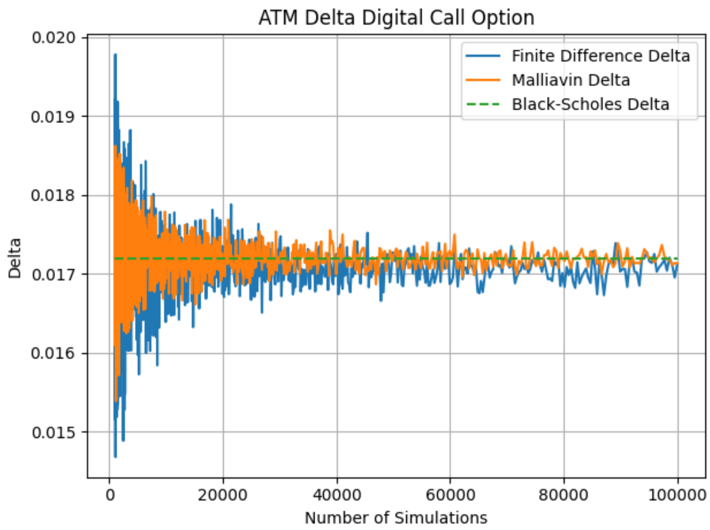

Figure 9: ATM Delta of a digital call option

In this case, the Malliavin method outperforms the finite difference approach. However, it becomes even clearer if we consider the Gamma of the digital call option:

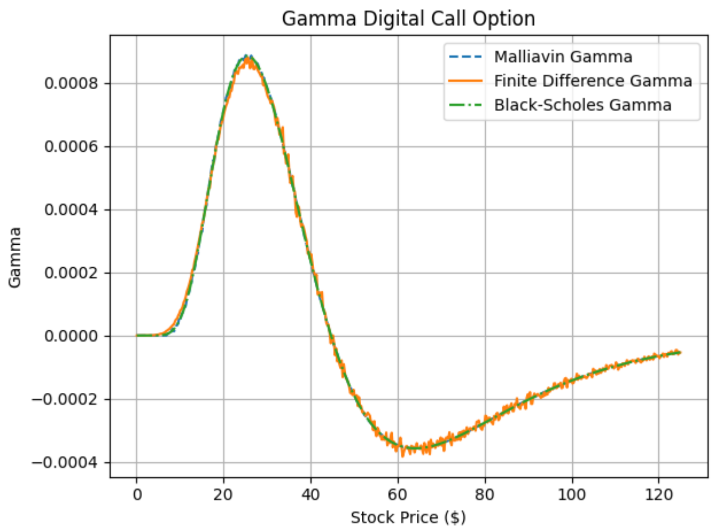

Figure 10: Gamma of a digital call option

At this stage, the finite difference method becomes unstable around the strike price and performs poorly. Finally, if we take a closer look at the strike, we find that the Malliavin approach outperforms the other method completely.

It is worth mentioning that in the finite difference method, we have kept the same step size value for all the calculations performed ($\Delta S = 5$). We have tried and chosen the value that minimizes the error.

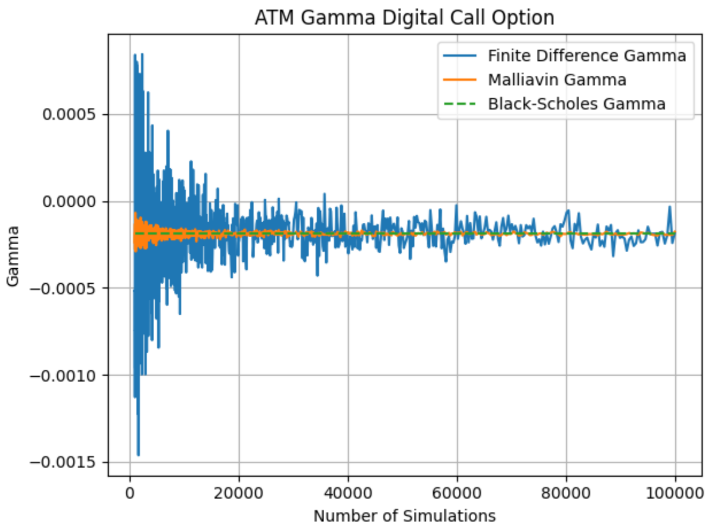

Figure 11: ATM Gamma of a digital call option

As a final note, we have used a digital option for comparison due to the fact that it has analytical solutions for the Greeks, making the analysis easier, as there is an exact solution that the two other methods should approach. However, it makes more sense to use the Malliavin tools for payoffs where a closed-form solution does not exist.

It is even possible to use the Malliavin techniques for other price dynamics that are different from Geometric Brownian Motion (such as the Heston Model, Fractional Brownian Motion...). For a deeper look at this, the reader can check [5].

Conclusion

We have successfully derived an analytic expression for the Delta and Gamma of any payoff function using the tools from Malliavin Calculus. Additionally, we have checked that this new approach for computing the Greeks outperforms the traditional finite difference methods for exotic options with discontinuous payoffs, especially for higher order Greeks.

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